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Linear MHD waves in Cartesian geometry. Transverse inhomogeneity and mixed properties.

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"This is why I love elementary school so much.
The kids really believe everything you tell them"
Principal Seymour Skinner to Mrs. Edna Crabapple
The Simpsons.

- **Linear motions superimposed on a static equilibrium state.**
- **Displace the plasma element from $\vec{r}$ to $\vec{r}+\vec{\xi}$**

$$\vec{r} \quad \rightarrow \quad \vec{r}+\vec{\xi}.$$

- $\vec{\xi} =$ ***Lagrangian displacement.***
- **Changes in density, pressure and magnetic field.**
- **Eulerian description / Lagrangian description.**
- **Linearise MHD equations.**

Equations for linear MHD waves

$$\rho' = -\nabla.(\rho_0\vec{\xi}),$$

$$p' = -\vec{\xi}.\nabla p_0 - \gamma p_0 \nabla.\vec{\xi},$$

$$\vec{B}' = \nabla \times \left(\vec{\xi} \times \vec{B}_0\right),$$

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -\nabla p' + \frac{1}{\mu}(\nabla \times \vec{B}_0) \times \vec{B}' + \frac{1}{\mu}(\nabla \times \vec{B}') \times \vec{B}_0$$

$$P' = p' + \frac{\vec{B}_0 \cdot \vec{B}'}{\mu}$$

- P' = **important quantity for understanding of mixed properties of MHD waves in non-uniform plasmas.**
- **Solutions of the form**

$$f(\vec{r};t) = \tilde{f}(\vec{r}) \exp(-i\omega t)$$

- $\tilde{f}(\vec{r})$ **is the time independent part of any of the perturbed quantities** f**.**

Linear MHD waves in Cartesian geometry

- **First step: Uniform plasma of infinite extent.**
- **Equilibrium quantities**

$$\vec{B}_0 = (0, 0, B_z),\ p_0 = \textbf{constant},\ \rho_0 = \textbf{constant}.$$

- **The coefficients of the partial differential equations in space are constants.**
- **Solutions in the form of plane waves**

$$\tilde{f}(\vec{r}) = \hat{f}\exp(i\vec{k}.\vec{r}) = \hat{f}\exp(i(k_x x + k_y y + k_z z)).$$

- **Combine the temporal and spatial factors**

$$\begin{aligned} f(\vec{r};t) &= \tilde{f}(\vec{r})\exp(-i\omega t) = \hat{f}\exp(i(\vec{k}.\vec{r} - \omega t)) \\ &= \hat{f}\exp(i(k_x x + k_y y + k_z z - \omega t)). \end{aligned}$$

- $\hat{f}$**= constant amplitude of** f**,** $\vec{k} = k_x\vec{1}_x + k_y\vec{1}_y + k_z\vec{1}_z$ **= the wave vector.**

- $\vec{B}_0$ **defines a preferred direction** : $\xi_z = \xi_\parallel, \quad (\nabla \times \vec{\xi})_z = (\nabla \times \vec{\xi})_\parallel$.

$$\begin{aligned} \xi_z &= \text{component parallel to } \vec{B}_0 \\ \nabla \cdot \vec{\xi} = i\,\vec{k} \cdot \vec{\xi} = i\,Y &= \text{compression} \\ (\nabla \times \vec{\xi})_z = i\,(\vec{k} \times \vec{\xi}) \cdot \vec{1}_z = i\,Z &= \text{component of vorticity parallel to } \vec{B}_0 \end{aligned}$$

- $X = k_z \xi_z, \quad Y = \vec{k} \cdot \vec{\xi}, \quad Z = (\vec{k} \times \vec{\xi}) \cdot \vec{1}_z$

$$\begin{aligned} \omega^2 X - k_z^2 v_\mathrm{s}^2 Y &= 0, \\ k^2 v_A^2\, X + (\omega^2 - k^2(v_A^2 + v_\mathrm{s}^2))Y &= 0, \\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet & \\ (\omega^2 - \omega_\mathrm{A}^2) Z &= 0. \end{aligned}$$

- **Two uncoupled subsets of equations.**
- **Two types of solutions.**

Classic Alfvén waves

- $\xi_z = 0, \;\; Y = 0, \;\; Z \neq 0.$

$$\omega^2 \; = \; \omega_A^2 = \frac{(\vec{k}\cdot\vec{B}_0)^2}{\mu\rho_0} \; = \; k_z^2 \, v_A^2, \;\; v_A^2 = \frac{B_0^2}{\mu \, \rho_0}$$

- ω_A = **local Alfvén frequency.**
- **No compression, no parallel displacement; parallel vorticity.**
- **Restoring force = Magnetic tension force.**
- **No variation of total pressure** $P' = 0$
- **Flow of energy along** $\vec{B}$ **with velocity** v_A**.**
- **Extremely an-isotropic.**
- **The displacement** $\vec{\xi}$ **for Alfvén waves .**

$$\vec{\xi}_A = (-\frac{k_y}{k_x}\vec{1}_x + \vec{1}_y)\xi_y \; = \; (\vec{1}_x - \frac{k_x}{k_y}\vec{1}_y)\xi_x.$$

- **Popular choice** $k_y = 0$ **:** $\vec{\xi}_A = \xi_y \vec{1}_y$**.**
- y**-independent Alfvén waves are a special case.**
- $k_y = 0 \Leftrightarrow m = 0, \;\; k_y \neq 0 \Leftrightarrow m \neq 0$**.**

- **Keep** $k_y \neq 0, \; k_z \neq 0$ **and finite.**
- **Take** $\lim k_x \to +\infty$ **so that** $\mid k_y \mid << \mid k_x \mid, \mid k_z \mid << \mid k_x \mid$.

$$\frac{\mid \xi_y \mid}{\mid \xi_x \mid} = \frac{\mid k_x \mid}{\mid k_y \mid} >> 1, \;\; \vec{\xi}_A \approx \xi_y \; \vec{1}_y.$$

- $\vec{\xi}$ **predominantly in the** y**-direction and rapidly varying in the** x**-direction.**
- **3 components of vorticity** $\nabla \times \vec{\xi}$ **are non-zero.**

$$(\nabla \times \vec{\xi})_z = i(k_x\xi_y - k_y\xi_x), \;\; (\nabla \times \vec{\xi})_x = -i \; k_z\xi_y, \;\; (\nabla \times \vec{\xi})_y = i \; k_z\xi_x, \;\; \xi_x = -\frac{k_y}{k_x}\xi_y.$$

- **Take** $\lim k_x \to +\infty$

$$\mid (\nabla \times \vec{\xi})_y \mid << \mid (\nabla \times \vec{\xi})_x \mid << \mid (\nabla \times \vec{\xi})_z \mid; \quad \nabla \times \vec{\xi} \approx (\nabla \times \vec{\xi})_z \; \vec{1}_z.$$

Magneto-sonic slow and fast waves

- $Y \neq 0,\ \xi_z \neq 0,\ Z = 0$.
- **Compression, parallel displacement, no parallel vorticity.**
- **Solutions**

$$\omega^2 = \omega_{sl,f}^2 = \frac{k^2(v_S^2 + v_A^2)}{2} \left\{ 1 \pm \left(1 - \frac{4\omega_C^2}{k^2(v_S^2+v_A^2)} \right)^{1/2} \right\}, \quad \omega^2_{(B=0)} = k^2 v_S^2$$

- ω_C = **the cusp frequency:** $\omega_C^2 = \dfrac{v_S^2}{v_S^2 + v_A^2}\omega_A^2, \quad v_S^2 = \dfrac{\gamma p_0}{\rho_0}$.
- **"sl" (slow) = the minus sign, "f" (fast) = the plus sign.**
- **Driven by tension and pressure forces.**
- **Variation of total pressure** $P' \neq 0$
- **Plasma pressure and magnetic pressure variations are in phase / antiphase.**
- **The displacement** $\vec{\xi}$ **for sl/f magneto-acoustic waves:**

$$\vec{\xi}_{sl,f} = (\vec{1}_x + \frac{k_y}{k_x}\,\vec{1}_y + \frac{\omega_{sl,f}^2 - k^2 v_A^2}{\omega_{sl,f}^2}\,\frac{k_z}{k_x}\vec{1}_z)\,\xi_x,$$

$$\vec{\xi}_{sl,f} = (\frac{\omega_{sl,f}^2}{\omega_{sl,f}^2 - k^2 v_A^2}\,\frac{k_x}{k_z}\vec{1}_x + \frac{\omega_{sl,f}^2}{\omega_{sl,f}^2 - k^2 v_A^2}\,\frac{k_y}{k_z}\vec{1}_y + \vec{1}_z)\xi_z.$$

- **Popular view: horizontal motion (ξ_x, ξ_y) is dominant for fast waves, parallel motion ξ_z is dominant for slow waves.**
- **True? Not the general rule!**
- **OK for strong magnetic fields, i.e.** $v_A >> v_S$

$$\vec{\xi}_f \approx (\vec{1}_x + \frac{k_y}{k_x}\,\vec{1}_y)\,\xi_x; \quad \vec{\xi}_{sl} \approx \xi_z \vec{1}_z.$$

- **No parallel vorticity but the horizontal components are non-zero**

$$\nabla \times \vec{\xi} = -i\,k_z\,\frac{k^2\,v_A^2}{\omega_{sl,f}^2}\,\xi_x\,(\frac{k_y}{k_x}\vec{1}_x - \vec{1}_y).$$

- **Division is clear.**
- **Parallel vorticity** $\neq\ 0\ \ \&$ **compression** $= 0,\ \ \xi_\| = 0$ **: Alfvén waves.**
- **Parallel vorticity** $=\ 0\ \ \&$ **compression** $\neq 0,\ \ \xi_\| \neq 0$ **: M-A waves.**
- **No mixing of properties.**
- **Pressureless plasma** $v_S^2 = 0: \ \ \omega_{sl}^2 = 0,\ \ \omega_f^2 = k^2 v_A^2,\ \ \xi_z = 0.$
- **No slow waves and the fast magneto-sonic waves have no parallel motions.**

- **Introduce non-uniformity.**

Surface Alfvén waves

- **Aim = study surface Alfvén waves on a density discontinuity**
- **Long way: study linear MHD waves in planar geometry**

Linear MHD waves in planar geometry

- **Cartesian coordinates** x, y, z.
- **Equilibrium model: Planar plasma in static equilibrium.**
- **Equilibrium quantities** $\vec{B}_0 = (0, 0, B_0((x)), \ p_0(x)$ **and** $\rho_0(x)$.
- **Two preferred directions:** $\vec{1}_x, \vec{1}_z$.
- $$\nabla \times \vec{B}_0 = -\frac{dB_0}{dx}\vec{1}_y, \ \ (\nabla \times \vec{B}_0) \times \vec{B}_0 = -\frac{d}{dx}(\frac{B_0^2}{2\mu})\vec{1}_x$$
- **Force balance equation**

$$\frac{d}{dx}(p_0 + \frac{B_0^2}{2\mu}) = 0$$

.

- **Fourier analyze with respect to the ignorable coordinates** y, z
- $$\exp(i(k_y y + k_z z)), \quad k^2 = k_y^2 + k_z^2.$$
- **Remember** $\exp(-i\omega t)$
- **Perturbed quantities** $\vec{\xi}, f'$ **are proportional to**

$$\exp(i(k_y y + k_z z - \omega t))$$

- $$\begin{aligned} f'(x, y, z; t) &= f'_\star(x) \exp(i(k_y y + k_z z - \omega t)) \\ \vec{\xi}(x, y, z, t) &= \vec{\xi}_\star(x) \exp(i(k_y y + k_z z - \omega t)) \end{aligned}$$
- **In what follows drop undersript** $\star$ **and drop factor** $\exp(i(k_y y + k_z z - \omega t)$
- $$\omega_A^2 = \frac{(k_z B_0(x))^2}{\mu \rho_0} = k_z^2 v_A^2(x)$$

- **$\omega_A(x)$ = local Alfvén frequency; $v_A(x)$ = local Alfvén velocity. In a non-uniform plasma $\omega_A(x)$ defines the Alfvën continuum.**
- $$[\textbf{min}\ \omega_A,\ \ \textbf{max}\ \omega_A]$$
- $$\omega_C^2 = \omega_A^2\,\frac{v_S^2}{v_S^2 + v_A^2}$$
- **$\omega_C(x)$ = local cusp frequency; $v_S(x)$ = local speed of sound. In a non-uniform plasma $\omega_C(x)$ defines the cusp or slow continuum.**
- $$[\textbf{min}\ \omega_C\ \ \textbf{max}\ \omega_C]$$
- **Aim = two 1st order ODE for ξ_x and P'**
- **Express the remaining variables in terms of ξ_x and P'**

- **Classic ODEs for ξ_x and P'**

$$D\frac{d\xi_x}{dx} = -C_2 P',$$
$$\frac{dP'}{dx} = \rho(\omega^2 - \omega_A^2)\xi_x.$$

$$D = \rho_0(v_S^2 + v_A^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2),$$
$$C_2 = \omega^4 - (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)\left(k_y^2 + k_z^2\right),$$

- **Equations for ξ_y, ξ_z, and $\nabla \cdot \vec{\xi}$**

$$\rho(\omega^2 - \omega_A^2)\xi_y = ik_y P',$$
$$\rho_0(\omega^2 - \omega_C^2)\xi_z = ik_z \frac{v_S^2}{v_S^2 + v_A^2} P'$$
$$\nabla \cdot \vec{\xi} = \frac{-\omega^2 \; P'}{\rho_0 \; (v_S^2 + v_A^2) \; (\omega^2 - \omega_C^2)}$$

- **Components of** $(\nabla \times \vec{\xi})$

$$
\begin{aligned}
(\nabla \times \xi) \cdot \vec{1}_x &= k_z\, k_y \frac{v_A^2}{v_S^2 + v_A^2} \frac{\omega^2}{\rho_0(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2)} P' \\
(\nabla \times \vec{\xi}) \cdot \vec{1}_y &= -ik_z \frac{d}{dx} \left\{ \frac{v_S^2}{v_A^2 + v_S^2} \frac{1}{\rho_0(\omega^2 - \omega_C^2)} \right\} P' \\
&+ ik_z \frac{\omega^2}{\rho_0(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2)} \frac{v_A^2}{v_A^2 + v_S^2} \frac{dP'}{dx} \\
(\nabla \times \vec{\xi}) \cdot \vec{1}_z &= -ik_y P' \frac{1}{\left\{\rho_0(\omega^2 - \omega_A^2)\right\}^2} \frac{d}{dx} \left\{\rho_0(\omega^2 - \omega_A^2)\right\}
\end{aligned}
$$

- **Note** v_S^2**,** v_A^2**,** ω_A^2**, and** ω_C^2 **are functions of position.**
- **The equations are coupled .**
- **The coupling functions** C_A **and** C_S
-

$$
C_A = ik_y P', \quad C_S = ik_z \frac{v_S^2}{v_S^2 + v_A^2} P'
$$

- **General rule: all wave variables are non-zero.**
- **No pure fast magneto-sonic waves and no pure Alfvén waves.**
- **Very different from infinite uniform plasma.**
- **A. Hasegawa and C. Uberoi, 1982 "The Alfvén wave".**

• **The basic characteristic of the ideal Alfvén wave is that the total pressure in the fluid remains constant during the passage of the wave as a consequence of the incompressibility condition. For inhomogeneous medium, however, the total pressure, in general, couples with the dynamics of the motion, and the assumption of neglect of pressure perturbations becomes invalid.**

- P' **couples the equations.**
- **The MHD waves have mixed properties.**
- **Always mixed properties except for** $k_y = 0$**:** $k_y = 0$ $C_A = 0$.
- **Equation for** ξ_y **is decoupled**

$$\rho(\omega^2 - \omega_A^2)\xi_y = 0$$

- **Pure Alfvén waves for** $k_y = 0$ **in a non-uniform planar plasma .**
- $k_y = 0$ **: y-invariant Alfvén waves and y-invariant magneto-sonic waves.**

Linear incompressible MHD waves in planar geometry

- **Incompressiblity means that the speed of sound is far faster than any other velocitiy in the system. Mathematically this means**
-

$$\nabla \cdot \vec{\xi} = 0, \quad \lim v_S \to \infty$$

- **Characteristic frequecies**

$$\omega_C = \omega_A$$

- **The slow and Alfvén continua coincide. Note that the Alfvén continuum has its dominant singularity in ξ_y while for the slow continuum the dominant singularity is in ξ_z. Both singularities are present.**
- **P' is a dependent unknown variable that cannot be computed by using expressions for the components of $\vec{\xi}$.**

- **Equations for incompressible motions on a non-uniform Cartesian 1-D equilbrium**

$$\begin{aligned} \rho(\omega^2-\omega_A^2)\frac{d}{dx}\left\{\frac{1}{\rho(\omega^2-\omega_A^2)}\frac{dP'}{dx}\right\} &= k^2\ P' \\ \rho(\omega^2-\omega_A^2)\xi_x &= \frac{dP'}{dx} \\ \rho(\omega^2-\omega_A^2)\xi_y &= ik_yP' \\ \rho_0(\omega^2-\omega_A^2)\xi_z &= ik_zP' \end{aligned}$$

- **Components of** $(\nabla\times\vec{\xi})$

$$\begin{aligned} (\nabla\times\vec{\xi})\cdot\vec{1}_x &= 0 \\ (\nabla\times\vec{\xi})\cdot\vec{1}_y &= ik_zP'\frac{1}{\{\rho_0(\omega^2-\omega_A^2)\}^2}\frac{d}{dx}\left\{\rho_0(\omega^2-\omega_A^2)\right\} \\ (\nabla\times\vec{\xi})\cdot\vec{1}_z &= -ik_yP'\frac{1}{\{\rho_0(\omega^2-\omega_A^2)\}^2}\frac{d}{dx}\left\{\rho_0(\omega^2-\omega_A^2)\right\} \end{aligned}$$

- **All wave variables, with the exception of compression and the $x-$component of $\nabla \times \vec{\xi}$, are non-zero in a non-uniform plasma . The 3 components of the Lagrangian displacement are non-zero.**

$$k^2 = k_y^2 + k_z^2$$

A piece-wise constant density planar geometry

$$\rho(x) = \begin{cases} \rho_i & \textbf{if } x \leq 0, \\ \rho_e & \textbf{if } x > 0. \end{cases}$$

- **Alfvén continuum is removed and replaced by 2 points:** $\omega_{A,i}, \omega_{A,e}$
- **2nd order ODE for** P'

$$\frac{d^2 P'}{dx^2} = k^2 P'$$

- **Solutions finite at** $\pm\infty$

$$P'_i(x) = A_1 \exp(+kx), \quad \xi_i(x) = A_1 \frac{k}{\rho_i(\omega^2 - \omega_{Ai}^2)} \exp(+kx)$$

$$P'_e(x) = A_2 \exp(-kx), \quad \xi_e(x) = A_2 \frac{-k}{\rho_e(\omega^2 - \omega_{Ae}^2)} \exp(-kx)$$

• **Continuity of P' and ξ_x at $x = 0$**

$$P_i'(0) \;=\; P_e'(0), \;\; \xi_i(0) = \xi_e(0)$$

results in

$$A_1 = A_2 = A$$

and the dispersion relation

$$\boxed{\rho_e(\omega^2 - \omega_{Ae}^2) + \rho_i(\omega^2 - \omega_{Ai}^2) = 0}$$

• **The dispersion relation can be solved for frequency ω in terms of k_z. This situation corresponds to standing waves with prescribed k_z and a corresponding solution for ω:**

$$\boxed{\omega^2 = \omega_k^2 \;=\; \frac{\rho_i\;\omega_{Ai}^2 + \rho_e\;\omega_{Ae}^2}{\rho_i + \rho_e} = k_z^2 v_k^2}$$

• **Conversely ω can be prescribed and the dispersion relation can be solved for k_z. This corresponds to propagating waves. The solution is :**

$$\boxed{k_z^2 = \frac{\omega^2}{v_k^2}}$$

v_k **is the kink speed. It is defined as**

$$v_k^2 = \frac{\rho_i\, \omega_{Ai}^2 + \rho_e\, \omega_{Ae}^2}{\rho_i + \rho_e}.$$

For constant magnetic field $B_{zi} = B_{ze}$

$$v_k^2 = \frac{2}{\rho_i + \rho_e} \frac{B_z^2}{\mu}$$

- $\xi_x(x), \xi_y(x), \xi_z(x), P'(x)$ **depend on** x**.**
- k **defines a length scale** $k = 1/R$**.**
-

$$\boxed{\exp(\pm(x/R)),\ x < 0 : +,\ \ x > 0 : - \quad R = 1/k}$$

- **Use constant** C

$$C = \frac{A}{(k_z R)^2 \frac{B_z^2}{\mu} \frac{\rho_i - \rho_e}{\rho_i + \rho_e}}$$

- **The solutions**

$$
\begin{aligned}
\frac{P_i'(x)}{(B^2/\mu)} &= C\ (k_z R)^2\ \frac{\rho_i - \rho_e}{\rho_i + \rho_e}\ \exp(x/R), \\
\frac{\xi_{x,i}(x)}{R} &= C\ \exp(x/R) \\
\frac{\xi_{y,i}(x)}{R} &= i\ C\alpha_y\ \exp(x/R) \\
\frac{\xi_{z,i}(x)}{R} &= i\ C\alpha_z\ \exp(x/R) \\
\frac{P_e'(x)}{(B^2/\mu)} &= C\ (k_z R)^2\ \frac{\rho_i - \rho_e}{\rho_i + \rho_e}\ \exp(-x/R), \\
\frac{\xi_{x,e}(x)}{R} &= C\ \exp(-x/R) \\
\frac{\xi_{y,e}(x)}{R} &= -i\ C\alpha_y\ \exp(-x/R) \\
\frac{\xi_{z,e}(x)}{R} &= -i\ C\alpha_z\ \exp(-x/R)
\end{aligned}
$$

$$
\alpha_y = \frac{k_y}{k},\ \alpha_z = \frac{k_z}{k}, \quad \frac{\xi_y}{\xi_z} = \frac{k_y}{k_z}
$$

- ξ_y, ξ_z **are discontinuous at** $x=0$ **due to change of sign of** $\omega^2-\omega_A^2$**.**

$$\xi_{y,e}(0) = -\xi_{y,i}(0),\ \xi_{z,e}(0) = -\xi_{z,i}(0)$$

- **Strong counterstreaming motions in the** $y-$ **and** $z-$ **directions. Possible cause of KH-instabilities.**
- **This effect is enhanced when the true discontinuity is replaced with a continuous variation.**
- **Strong shear at** $x=0$
- **Vorticity is present at** $x=0$ **due to discontinuity in** ξ_y **and** ξ_z**.**

$$\begin{aligned}
(\nabla \times \vec{\xi}) \cdot \vec{1}_x &= 0 \\
(\nabla \times \vec{\xi}) \cdot \vec{1}_y &= -2k_z\, P' \,\frac{\rho_i+\rho_e}{\rho_e\, \rho_i}\, \frac{1}{\omega_{A,e}^2-\omega_{A,i}^2}\ \delta(x) \\
(\nabla \times \vec{\xi}) \cdot \vec{1}_z &= 2k_y\, P' \,\frac{\rho_i+\rho_e}{\rho_e\, \rho_i}\, \frac{1}{\omega_{A,e}^2-\omega_{A,i}^2}\ \delta(x)
\end{aligned}$$

Resonant damping

• **The true discontinuity at the surface $x = 0$ is replaced by a continuous variation from ρ_i to ρ_e in an intermediate layer of thickness l $[-l/2 \;\; l/2]$.**

• **The characteristic frequencies $\omega_A^2(x) = \omega_C^2(x)$ vary with position x and define the continuous spectrum of resonant Alfvén (slow) waves. Here the two continua coincide.**

Alfvén continuous spectrum $= [\textbf{min}\, \omega_A(x), \;\; \textbf{max}\, \omega_A(x)\,]$

• **Take density to be a monotically decreasing function in the non-uniform layer so that**

$$\textbf{min}\, \omega_A(x) = \omega_{A,i}, \quad \textbf{max}\, \omega_A(x) = \omega_{A,e}$$

Obviously

$$\omega_{A,i} < \omega_k < \omega_{A,e}$$

so the surface Alfvén / slow wave is resonantly damped.

• **No long wave length (thin tube) approximation!**

• **Adopt the thin boundary approximation and use the jump condition for incompressible motions:**

$$[P'] = 0, \quad [\xi_x] = -i\pi \frac{1}{\rho(x_A) \mid \Delta \mid}(k_y^2 + k_z^2)P' \;=\; -i\pi \frac{1}{\rho(x_A) \mid \Delta \mid} k^2 P'$$

- **Role of P' Hollweg and Yang 1988, JGR, 93, 93, 5423 - 5436 Resonance absorption can occur in any situation where total pressure fluctuations are imparted to field lines satisfying the Alfvén and cusp resonances conditions.**
- **Jump in flux of energy accross resonant position is proportioal to $\mid P' \mid^2$.**
- **x_A is the position of the resonance where $\omega_k = \omega_A(x_A)$. In the thin boundary approximation $x_A = 0$. The quantity Δ is**

$$\Delta = \frac{d}{dx} \left\{ \omega^2 - \omega_A^2 \right\} \mid_{x_A}$$

- **The condition on ξ_x is then**

$$1 + F - iG = 0$$

with

$$F = \frac{\rho_e(\omega^2 - \omega_{A,e}^2)}{\rho_i(\omega^2 - \omega_{A,i}^2)}, \quad G = \pi \frac{\rho_e(\omega^2 - \omega_{A,e}^2)}{\rho(x_A) \mid \Delta \mid} \; k$$

- **The term $-iG$ contains the effect of the resonant damping. When we put $G = 0$ we recover the dispersion relation for the true discontinuity. Its solution is $\omega = \omega_R = \omega_k$.**
- **Rewrite the dispersion relation $1 + F - iG = 0$ as**

$$\omega^2 - k_z^2 v_k^2 = \frac{1}{\rho_i + \rho_e} \, \frac{i\pi \; k}{\rho(x_A) \mid \Delta \mid} \; \rho_i(\omega^2 - \omega_{A,i}^2) \; \rho_e(\omega^2 - \omega_{A,e}^2)$$

- **Because of resonant damping the frequency is complex.**

$$\omega = \omega_R + i\gamma$$

- **The period and the damping time τ_D are**

$$\textbf{Period} = \frac{2\pi}{\omega_R}, \quad \tau_D = \frac{1}{\mid \gamma \mid}$$

- **Weak damping: use a perturbation method. Neglect the effect of resonant damping on the real part of the frequency so that $\omega_R \approx \omega_k$. Also neglect terms of second order in γ/ω_R . Hence**

$$\omega_R \approx \omega_k = \omega_A(x_A), \quad \omega^2 \approx \omega_R^2 + \ 2i\omega_R\,\gamma$$

- **Find then**

$$\frac{\gamma}{\omega_R} \approx \frac{-\pi}{2\ \omega_k^2}\,\frac{\rho_i^2\ \rho_e^2}{(\rho_i+\rho_e)^3}\,\frac{(\omega_{A,i}^2-\omega_{A,e}^2)^2}{\rho(x_A)\mid \Delta \mid}\ k$$

- **Take $k = 1/R$**

$$\frac{\gamma}{\omega_R} \approx \frac{-\pi}{2\omega_k^2}\,\frac{\rho_i^2\ \rho_e^2}{(\rho_i+\rho_e)^3}\,\frac{(\omega_{A,i}^2-\omega_{A,e}^2)^2}{\rho(x_A)\mid \Delta \mid}\,\frac{1}{R}$$

This is exactly Equation 77 of GHS92 when this equation is corrected for a typo and $\mid m \mid = 1$.

- **Now take equal and constant magnetic fields $B_1 = B_2 = B$**
- **Straightforward calculation leads to**

$$\gamma = -\frac{\pi}{2R} \frac{\rho_A \mid \omega_k \mid^3}{\mid \Delta \mid} \frac{(\rho_2 - \rho_1)^2}{(\rho_1 + \rho_2)^3}.$$

This is exactly Equation 56 of Ruderman and Roberts, 2002

When $B_0 =$ constant it follows that

$$\rho_0(x_A) \mid \Delta \mid = \omega_A^2(x_A) \mid \frac{d\rho_0}{dx} \mid_{x_A} = \omega_k^2 \mid \frac{d\rho_0}{dx} \mid_{x_A}$$

- **Hence**

$$\frac{\gamma}{\omega_R} = -\frac{\pi}{8} \frac{(\rho_i - \rho_e)^2}{\rho_i + \rho_e} k \frac{1}{\mid \frac{d\rho(x)}{dx} \mid_{x_A}}$$

Replace k with $k = 1/R$ then

$$\frac{\gamma}{\omega_R} = -\frac{\pi}{8} \frac{(\rho_i - \rho_e)^2}{\rho_i + \rho_e} \frac{1}{R} \frac{1}{\left| \frac{d\rho(x)}{dx} \right|_{x_A}}$$

• **Compare this expression for γ/ω_R with those derived in Goossens et al. 2009. See their equations 31 for a pressureless compressible cylindrical plasma and equation 53 for an incompressible cylindrical plasma. Equations 31 and 53 are derived in the long wave length limit $(k_z R << 1)$. Equation 31 is identical to the equation derived here. The same applies to equation 53 when we drop the quadratic terms in $k_z R$. This is a remarkable result. Cylindrical or planar geometry does not make a difference in the long wave length limit.**

• **The damping time τ_D**

$$\frac{\tau_D}{\textbf{Period}} = \frac{4}{\pi^2} \frac{\rho_i + \rho_e}{(\rho_i - \rho_e)^2} \frac{1}{k} \left| \frac{d\rho(x)}{dx} \right|_{x_A}$$

- **Assume that density $\rho_0(x)$ varies from ρ_i at $x = -l/2$ to ρ_e at $x = +l/2$ with steepness α so that**

$$\frac{d\rho(x)}{dx} = -\alpha \frac{\rho_i - \rho_e}{l}$$

For a linear profile $\alpha = 1$, for a sinusoidal profile $\alpha = \pi/2$.

- **Damping decrement $\frac{\gamma}{\omega_R}$**

$$\frac{\gamma}{\omega_R} = -\frac{\pi}{8} \frac{(\rho_i - \rho_e)}{\rho_i + \rho_e} \frac{kl}{\alpha} = -\frac{\pi}{8} \frac{(\rho_i - \rho_e)}{\rho_i + \rho_e} \frac{l/R}{\alpha}$$

- **The damping time τ_D**

$$\frac{\tau_D}{\textbf{Period}} = \frac{4}{\pi^2} \frac{\rho_i + \rho_e}{\rho_i - \rho_e} \frac{\alpha}{kl} = \frac{4}{\pi^2} \frac{\rho_i + \rho_e}{\rho_i - \rho_e} \frac{\alpha}{l/R}$$

- **The expressions for γ/ω_R and τ_D/Period agree with the corresponding expressions listed in Goossens et al. 2009 for a cylindrical plasma in the thin tube approximation.**

• **Go back to ξ_y, ξ_z and components of $(\nabla \times \vec{\xi})$**

$$\begin{aligned}
\xi_y &= i k_y P' \frac{1}{\rho_0(\omega^2 - \omega_A^2)} \\
\xi_z &= i k_z P' \frac{1}{\rho_0(\omega^2 - \omega_A^2)}
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \vec{\xi}) \cdot \vec{1}_y &= i k_z P' \frac{1}{\{\rho_0(\omega^2 - \omega_A^2)\}^2} \frac{d}{dx} \left\{\rho_0(\omega^2 - \omega_A^2)\right\} \\
(\nabla \times \vec{\xi}) \cdot \vec{1}_z &= -i k_y P' \frac{1}{\{\rho_0(\omega^2 - \omega_A^2)\}^2} \frac{d}{dx} \left\{\rho_0(\omega^2 - \omega_A^2)\right\}
\end{aligned}$$

• **Recall that for MHD waves (weakly) resonantly damped in the Alfvén continuum**

$$\omega = \omega_R + i\gamma, \quad \omega_R \approx \omega_A(x_A), \quad \gamma << \omega_R$$

• **The values of ξ_y, ξ_z are not infinte but very large in absolute value with opposite values of their real parts across the ideal resonant point $x = x_A$.**

• **KH-instabilities are difficult to avoid.**

• Propagating waves. Now the frequency is given and the dispersion relation determines the wave number k_z.

• Follow and Terradas, Goossens and Verth 2010 (TGV 2010 , A&A, 524, A23) in case of a cylindrical plasma with a straight field.

• No need to use the long wave limit or the thin tube approximation.

• The wave number k_z is complex

$$k_z = k_{z,R} + ik_{z,I}$$

• The wave length λ and the damping length L_D are

$$\lambda = \frac{2\pi}{k_{z,R}}, \quad L_D = \frac{1}{\mid k_{z,I} \mid}$$

• Use a perturbation method. Neglect the effect of resonant damping on the real part of the wave number so that

$$k_{z,R}^2 \approx \frac{\omega^2}{v_k^2}$$

• The result for the damping lenght L_D

$$\boxed{\frac{L_D}{\lambda} = \frac{4}{\pi^2} \frac{\rho_i + \rho_e}{(\rho_i - \rho_e)^2} \frac{1}{k_0} \mid \frac{d\rho(x)}{dx} \mid_{x_A}}$$

-

$$k_0 = \sqrt{k_y^2 + k_{z,R}^2}$$

- **As before replace the derivative of equilibrium density with**

$$\frac{d\rho(x)}{dx} = -\alpha \frac{\rho_i - \rho_e}{l}$$

and arrive at

$$\boxed{\frac{L_D}{\lambda} = \frac{4}{\pi^2} \frac{\rho_i + \rho_e}{\rho_i - \rho_e} \frac{\alpha}{k_0\, l}}$$

Conclusions & Comments

- **Linear MHD : No interaction of MHD waves.**
- **Given wave is defined in the whole space.**
- **Non-uniformity across magnetic field is fundamental.**
- **It generates MHD waves with mixed properties.**
- **Pure Alfvén waves and pure magneto-acoustic waves ???**
- **Non-zero pressure variations & non-zero horizontal and parallel vorticity.**
- **Properties of given wave depend on properties of background.**
- **Resonant absorption in Alfvén continuum. Total pressure perturbation is non-zero.**
- **≈ Alfvén wave = Alfvénic wave .**
- **Mixed properties = general phenomenon. It does not depend on geometry. Non-uniformity across magnetic field is key.**